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Existence and regularity of a weak function-solution for some Landau equations with a stochastic approach

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Abstract

Using the Malliavin Calculus, this paper proves the existence of a weak function-solution of class $\mathscr{C}^{\infty}$ with bounded derivatives of the Landau equation for a generalization of Maxwellian molecules when the initial data is a probability measure. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Landau equation, also called the Fokker–Planck–Landau equation, is obtained as limit of the Boltzmann equation when all the collisions become grazing. Its expression, in the spatially homogeneous case, is

$$\frac{\partial f}{\partial t}(v,t)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d}\mathrm{d}v_*\,a_{ij}(v-v_*)\left[f(v_*,t)\,\frac{\partial f}{\partial v_j}(v,t)-f(v,t)\,\frac{\partial f}{\partial v_{*j}}(v_*,t)\right]\right\},\tag{1}$$

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where $f(v,t) \geqslant 0$ is the density of particles with velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$, and $(a_{ij}(z))_{1 \leqslant i,j \leqslant d}$ is a nonnegative symmetric matrix depending on the interaction between the particles.

In this paper, we study the Landau equation for a generalization of Maxwell gas. We consider a matrix a of the form

$$a_{ij}(z) = h(|z|^2)(|z|^2\delta_{ij} - z_i z_j), \tag{2}$$

where h is a positive continuous function on $\mathbb{R}_+$ such that there exist $m, M > 0$ with $\forall z \in \mathbb{R}^d$

$$m \leqslant h(|z|^2) \leqslant M. \tag{3}$$

When h is a constant, we recognize the coefficient of the Landau equation for Maxwellian molecules.

We define the vector b by

$$b_i(z) = \sum_{j=1}^{d} \partial_j a_{ij}(z) = -(d-1)h(|z|^2)z_i. \tag{4}$$

Then, by integration by parts, we can give a weak formulation of Eq. (1), and consequently we define the notion of weak function-solution.

Definition 1. Let $f(.,0)$ be a nonnegative function on $\mathbb{R}^d$ with finite mass and energy. A nonnegative function f on $\mathbb{R}^d \times \mathbb{R}^+$ is a weak function-solution of the Landau equation with initial data $f(.,0)$, if f satisfies the following equation for any test function $\varphi \in \mathscr{C}_b^2(\mathbb{R}^d, \mathbb{R})$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi(v) f(v,t)\,\mathrm{d}v \\
&\quad = \frac{1}{4} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}v\,\mathrm{d}v_* f(v,t) f(v_*,t) a_{ij}(v - v_*)(\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*)) \\
&\qquad + \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}v\,\mathrm{d}v_* f(v,t) f(v_*,t) b_i(v - v_*)(\partial_i\varphi(v) - \partial_i\varphi(v_*)),
\end{aligned} \tag{5}$$

where $\partial_i \varphi = \partial\varphi/\partial v_i$ and $\partial_{ij}\varphi = \partial^2\varphi/\partial v_i \partial v_j$.

Eq. (5) conserves mass, momentum and energy. Thus, if there exists a weak function-solution f of (5) with an initial data satisfying $\int_{\mathbb{R}^d} f(v,0)\,\mathrm{d}v = 1$, the measure P_t on $\mathbb{R}^d$ given by $P_t(\mathrm{d}v) = f(v,t)\,\mathrm{d}v$ is a probability measure, for any $t \geqslant 0$. Thus, we define a probabilistic notion of solutions of the Landau equation.

Definition 2. Let P_0 be a probability measure on $\mathbb{R}^d$ with a finite two-order moment (i.e. $\int_{\mathbb{R}^d} |v|^2 P_0(\mathrm{d}v) < \infty$). A *measure-solution* of the Landau equation (6) with initial data P_0 is a probability flow $(P_t)_{t\geqslant 0}$ on $\mathbb{R}^d$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(v)P_t(\mathrm{d}v) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} P_t(\mathrm{d}v)\left(\int_{\mathbb{R}^d} P_t(\mathrm{d}v_*)a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v)$$
$$+\sum_{i=1}^{d}\int_{\mathbb{R}^d} P_t(\mathrm{d}v)\left(\int_{\mathbb{R}^d} P_t(\mathrm{d}v_*)b_i(v-v_*)\right)\partial_i\varphi(v) \tag{6}$$

for any function $\varphi\in\mathscr{C}_b^2(\mathbb{R}^d,\mathbb{R})$.

This approach allows us to have weaker conditions on the initial data, i.e. we can assume that the initial data is a probability measure and not necessarily a density of probability.

Remark 3. *With an abuse of notation, we will still say that a probability measure P on $\mathscr{C}([0,T],\mathbb{R}^d)$ is a measure-solution of the Landau equation when its time-marginals flow is a measure-solution in the sense of Definition* 2.

Guérin (2000) has already stated the existence of a probability measure-solution of Landau equation (6). We are here interested in proving with a stochastic approach the existence of a bounded weak function-solution of (5) of class $\mathscr{C}^\infty$ with bounded derivatives.

Let us briefly recall the main results of Guérin (2000). We will associate with Landau equation (6) a nonlinear stochastic differential equation driven by a space–time white noise. We highlight the nonlinearity using two probability spaces: $(\Omega,\mathscr{F},\mathbb{P})$ and $([0,1],\mathscr{B}([0,1]),\mathrm{d}\alpha)$, where $\mathrm{d}\alpha$ is the Lebesgue measure on $[0,1]$. In order to avoid any confusion, we will denote by E the expectation and $\mathscr{L}$ the distribution of a random variable on $(\Omega,\mathscr{F},\mathbb{P})$ and E_α, $\mathscr{L}_\alpha$ for a random variable on $([0,1],\mathscr{B}([0,1]),\mathrm{d}\alpha)$.

Since a is a nonnegative symmetric matrix, there exists a matrix σ of order $d\times d'$ such that

$$a=\sigma.\sigma^*, \tag{7}$$

where σ^* is the adjoint matrix of σ.

We define a d'-dimensional space–time white noise on $[0,1]\times[0,\infty)$, by

$$W^{d'} = \begin{pmatrix} W_1 \\ \vdots \\ W_{d'} \end{pmatrix}, \tag{8}$$

where the W_i are independent space–time white noises with covariance measure $\mathrm{d}\alpha\,\mathrm{d}t$ on $[0,1]\times[0,\infty)$ (according to the definition of Walsh (1984)). We consider its natural filtration $(\mathscr{F}_t)_{t\geqslant 0}$, $\mathscr{F}_t=\sigma\{W^{d'}([0,s]\times A)\colon 0\leqslant s\leqslant t,\ A\in\mathscr{B}([0,1])\}$.

For $k \geqslant 2$, we define $\mathscr{P}_k$ the space of continuous adapted processes $X = (X_t)_{t \geqslant 0}$ from $(\Omega, F, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$ to $\mathbb{R}^d$, such that

$$E\left[\sup_{0 \leqslant t \leqslant T} |X_t|^k\right] < \infty$$

for any $T > 0$, and $\mathscr{P}_{k,\alpha}$ the space of continuous processes $Y = (Y_t)_{t \geqslant 0}$ from $([0,1], \mathscr{B}([0,1]), \mathrm{d}\alpha)$ to $\mathbb{R}^d$, such that

$$E_\alpha\left[\sup_{0 \leqslant t \leqslant T} |Y_t|^k\right] < \infty$$

for any $T > 0$.

Let X_0 be a random vector on $\mathbb{R}^d$, independent of $W^{d'}$, with a finite moment of order 2.

We consider the following nonlinear stochastic differential equation.

Definition 4. Let X_0 and $W^{d'}$ be defined as below. A couple of processes (X, Y) on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P}) \times ([0,1], \mathscr{B}([0,1]), \mathrm{d}\alpha)$ is solution of the nonlinear stochastic differential equation (NSDE(σ, b)) if for any $t \geqslant 0$

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)).W^{d'}(\mathrm{d}\alpha, \mathrm{d}s) + \int_0^t \int_0^1 b(X_s - Y_s(\alpha))\, \mathrm{d}\alpha\, \mathrm{d}s$$

and $\mathscr{L}(X) = \mathscr{L}_\alpha(Y)$.

We notice, using Ito's Formula, that the distribution of a solution of (NSDE(σ, b)) is a weak measure-solution of the Landau equation (6) with initial data $P_0 = \mathscr{L}(X_0)$.

In Guérin (2000) (Theorem 10), we have proved the following theorem for $k = 2$, but, adapting the proofs, it is still true for any $k \geqslant 2$.

Theorem 5. *Assume that $W^{d'}$ is a d'-dimensional space–time white noise and assume that X_0 is an independent random vector on $\mathbb{R}^d$ with finite moment of order k. If the functions σ and b, defined by* (7), (2) *and* (4), *are Lipschitz continuous, there exists a couple (X, Y), unique in law, solution of the nonlinear equation* (NSDE(σ, b)) *with* $(X, Y) \in \mathscr{P}_k \times \mathscr{P}_{k,\alpha}$.

Corollary 6. *Assume that P_0 is a probability measure with a finite moment of order* 2. *There exists a measure-solution $(P_t)_{t \geqslant 0}$ with initial data P_0 to Landau equation* (6) *when σ and b are Lipschitz continuous functions.*

Corollary 7. *Assume that P_0 is a probability measure with a finite moment of order* 2. *There is uniqueness of the measure-solution $(P_t)_{t \geqslant 0}$ with initial data P_0 to Landau equation* (6) *when σ and b are Lipschitz continuous functions.*

Proof.

- We just have to prove the uniqueness of the solution $(Q_t^{\mu})_{t\geqslant 0}$ of the linear Landau equation, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(v)Q_t(\mathrm{d}v)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}Q_t(\mathrm{d}v)\left(\int_{\mathbb{R}^d}\mu_t(\mathrm{d}v_*)a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v)$$
$$+\sum_{i=1}^{d}\int_{\mathbb{R}^d}Q_t(\mathrm{d}v)\left(\int_{\mathbb{R}^d}\mu_t(\mathrm{d}v_*)b_i(v-v_*)\right)\partial_i\varphi(v), \qquad (9)$$

  where $(\mu_t)_{t\geqslant 0}$ is a probability flow on $\mathbb{R}^d$. Indeed, as a measure-solution $(P_t)_{t\geqslant 0}$ to the (*nonlinear*) Landau equation is also a solution of the linear equation (9) with $\mu_t=P_t$ for any $t\geqslant 0$, by uniqueness of the measure-solution of (9), $(P_t)_{t\geqslant 0}$ is uniquely determined.
- The linear Landau equation (9) satisfies the assumptions of Theorem 5.2 in Bhatt and Karandikar (1993), then there is uniqueness of the measure-solution of (9). □

Remark 8. *We notice that we can choose*

$$\sigma(z)=\sqrt{h(|z|^2)}\begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \qquad (10)$$

in dimension two, and

$$\sigma(z)=\sqrt{h(|z|^2)}\begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix} \qquad (11)$$

in dimension three. Then, if h is a bounded function of class $\mathscr{C}^1$ with $h'(x)=\mathrm{O}(1/x^2)$ when $x\to+\infty$, σ and b are Lipschitz continuous functions of class $\mathscr{C}^1$ on $\mathbb{R}^d$, for $d=2,3$. This property can be generalized in higher dimension.

The aim of this article is to find a weak function-solution of the Landau equation when the initial data is a probability measure. To state the existence of a weak function-solution of (5) from a measure-solution, it is enough to show that the measure-solution is absolutely continuous with respect to the Lebesgue measure. Indeed, if $(P_t)_{t\geqslant 0}$ is a measure-solution of (6) with initial data P_0 and if there exists a nonnegative function f_t on $\mathbb{R}^d$ such that $P_t(\mathrm{d}v)=f_t(v)\,\mathrm{d}v$ for any $t>0$, then the function f defined by $f(v,t)=f_t(v)$ for any $v\in\mathbb{R}^d$, $t>0$, is a weak function-solution of (5) with initial data P_0.

The idea consists in using the relation between the Landau equation and the nonlinear differential equation (NSDE(σ,b)). In fact, we develop a Malliavin Calculus for the value X_t, $t>0$, of the solution X of (NSDE(σ,b)) obtained in Theorem 5, inspired by the methods used by Bally and Pardoux (1998) and Nualart (1995).

The Maxwellian case (i.e., when the function h is a constant) is studied in detail with an analytic approach by Villani (1998). When the initial data is a nonnegative function f_0 with finite mass and energy, Villani has proved the existence and the uniqueness of a bounded solution of (1) of class $\mathscr{C}^\infty$.

We prove here the existence of a bounded weak function-solution of Landau equation (5) of class $\mathscr{C}^\infty$ with bounded derivatives when the initial data is a probability measure with finite moments for some bounded functions h.

1.1. About the Malliavin calculus for a white noise

The Malliavin calculus is almost the same for the white noise as for the Brownian Motion. We use in the following the same notation as Nualart (1995). We just recall the definitions of the main spaces for the Malliavin calculus.

Let $W^{d'}$ be a d'-dimensional space–time white noise.

Let $\mathscr{S}$ be the class of random variables F having the following form:

$$F = f(W^{d'}(g_1), \dots, W^{d'}(g_n)),$$

where f is a $\mathscr{C}^\infty$ real valued function on $\mathbb{R}^n$ with partial derivatives having polynomial growth, $g = (g_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant d'}}$ is a matrix with components in $\mathbb{L}^2([0,1]\times[0,\infty), \mathrm{d}\alpha\,\mathrm{d}s)$, and

$$W^{d'}(g_i) = \sum_{j=1}^{d'} \int_0^\infty \int_0^1 g_{ij}(s,\alpha) W_j(\mathrm{d}\alpha, \mathrm{d}s).$$

Assuming that $(r,z)\in[0,\infty)\times[0,1]$, we define the first-order Malliavin derivative $D^l_{(r,z)}F$ of F in relation to the lth white noise W_l at point (r,z), with $l\in\{1,\dots,d'\}$, by

$$D^l_{(r,z)}F = \sum_{i=1}^n \partial_i f(W^{d'}(g_1), \dots, W^{d'}(g_n)) g_{il}(r,z).$$

For $k\geqslant 1$, set $\lambda_k = ((r_1,z_1),\dots,(r_k,z_k))$ with $r_m\in[0,\infty)$ and $z_m\in[0,1]$, $m=1,\dots,k$. We define by iteration the derivatives of order k. Let $(l_1,\dots,l_k)$ be a k-uplet of $\{1,\dots,d'\}$, we denote by

$$D^{l_1,\dots,l_k}_{\lambda_k}F = D^{l_k}_{(r_k,z_k)} D^{l_{k-1}}_{(r_{k-1},z_{k-1})} \cdots D^{l_1}_{(r_1,z_1)}F.$$

We denote by $\mathbb{D}^{k,p}$ the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{m=1}^k \sum_{l_1,\dots,l_m=1}^{d'} E(\|D^{l_1,\dots,l_m}F\|^p_{\mathbb{L}^2(\Lambda_m)}) \right]^{1/p},$$

where

$$\|D^{l_1,\dots,l_m}F\|^2_{\mathbb{L}^2(\Lambda_m)} = \int_{\Lambda_m} |D^{l_1,\dots,l_m}_{\lambda_m}F|^2\,\mathrm{d}\lambda_m$$

with $\Lambda_m = ([0,\infty)\times[0,1])^m$ and $\lambda_m = ((r_1,z_1),\dots,(r_m,z_m))\in\Lambda_m$.

We also denote by $\mathbb{D}^\infty$ the subspace of the infinitely differentiable variables:

$$\mathbb{D}^\infty = \bigcap_{p\geqslant 1}\bigcap_{k\geqslant 1}\mathbb{D}^{k,p},$$

When F is a random vector in $\mathbb{R}^d$, we differentiate component by component and we denote by DF the matrix $(DF)_{i,l} = D^l F_i$, $1 \leqslant i \leqslant d$, $1 \leqslant l \leqslant d'$. The Malliavin matrix is defined by

$$I = \int_0^T \int_0^1 D_{(r,z)}F.(D_{(r,z)}F)^* \, \mathrm{d}z \, \mathrm{d}r.$$

In this paper, under suitable assumptions on σ and b and integrability conditions on the initial data X_0, we show that for any $t > 0$ the value X_t of X obtained in Theorem 5 satisfies the conditions of one of those two following theorems.

Theorem (a) (see Nualart, 1995, Theorem 2.1.2).
Let $F = (F_1, \ldots, F_d)$ be a random vector on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying the following conditions:
(i) *F_i belongs to the space $\mathbb{D}^{1,p}$, $p > 1$, for any $i = 1, \ldots, d$.*
(ii) *The Malliavin matrix $I = \int_0^\infty \int_0^1 D_{(r,z)}F.(D_{(r,z)}F)^* \, \mathrm{d}z \, \mathrm{d}r$ is invertible a.s.*
Then the distribution of F is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^d$.

Theorem (b) (see Nualart, 1995, Corollary 2.1.2).
Let $F = (F_1, \ldots, F_d)$ be a random vector on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying the following conditions:
(i) *F_i belongs to $\mathbb{D}^\infty$, for any $i = 1, \ldots, d$.*
(ii) *The Malliavin matrix $I = \int_0^\infty \int_0^1 D_{(r,z)}F.(D_{(r,z)}F)^* \, \mathrm{d}z \, \mathrm{d}r$ satisfies*

$$(\det I)^{-1} \in \bigcap_{p>1} \mathbb{L}^p(\Omega).$$

Then F has an infinitely differentiable density.

1.2. Notations

- $\mathscr{C}([0,T], \mathbb{R}^d)$ is the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and for $k \in \mathbb{N}$, $\mathscr{C}_b^k([0,T], \mathbb{R}^d)$ is the space of functions of class $\mathscr{C}^k$ with all its derivatives bounded up to order k.
- $\mathscr{M}_{d,d'}(\mathbb{R})$ is the set of $d \times d'$ matrix on $\mathbb{R}$.
- For $k \geqslant 2$, a random variable Z on $(\Omega, \mathscr{F}, \mathbb{P})$ belongs to $\mathbb{L}^k$ if Z has a finite moment of order k, i.e. $E[|Z|^k] < \infty$.
- K is an arbitrary notation for a positive constant (K can change from line to line).

2. Computation of the derivatives of X

2.1. The first derivative

Assumption (H^1). σ and b are Lipschitz continuous functions of class $\mathscr{C}^1$ from $\mathbb{R}^d$ to $\mathscr{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$, respectively.

We denote by K_σ and K_b their Lipschitz constants.

Theorem 9. *We assume that X_0 has a finite 2-order moment. Let (X,Y) be the solution of the nonlinear stochastic differential equation* (NSDE(σ,b)) *obtained in Theorem* 5. (Y *will play a parameter role in the following.*)

Under Assumption (H^1), $\forall t\in[0,T]$ $\forall i=1,\dots,d$, $X_{i,t}\in\mathbb{D}^{1,2}$. *The* ith *component of its derivative in relation to the* lth *white noise at point* $(r,z)\in[0,\infty)\times[0,1]$ *is given by*

$$D^l_{(r,z)}X_{i,t}=\sigma_{i,l}(X_r-Y_r(z))+\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}(X_s-Y_s(\alpha))D^l_{(r,z)}X_{m,s}W_k\,(\mathrm{d}\alpha,\mathrm{d}s)$$

$$+\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i(X_s-Y_s(\alpha))D^l_{(r,z)}X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s$$

if $t\geqslant r$ *and* $D^l_{(r,z)}X_{i,t}=0$ *if* $t<r$.

Proof. Since X_0 is independant of the white noise, we notice that we can extend the definition of the sets $\mathbb{D}^{k,p}$ and $\mathbb{D}^\infty$ to the case we are considering.

We consider the Picard sequence of $\mathscr{P}_2$-processes defined by

$$X_t^0=X_0$$

$$X_t^{n+1}=X_0+\int_0^t\int_0^1\sigma(X_s^n-Y_s(\alpha))W^{d'}(\mathrm{d}\alpha,\mathrm{d}s)$$

$$+\int_0^t\int_0^1 b(X_s^n-Y_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s. \tag{12}$$

Then, the ith component writes

$$X_{i,t}^{n+1}=X_{i,0}+\int_0^t\int_0^1\sum_{k=1}^{d'}\sigma_{i,k}(X_s^n-Y_s(\alpha))W_k\,(\mathrm{d}\alpha,\mathrm{d}s)$$

$$+\int_0^t\int_0^1 b_i(X_s^n-Y_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s.$$

According to Guérin (2000) Theorem 8, the sequence (X^n) satisfies

$$\sup_n E\left[\sup_{0\leqslant r\leqslant T}|X_r^n|^2\right]<\infty \tag{13}$$

and converges for the norm

$$\|U\| = \left\| \sup_{0 \leqslant t \leqslant T} |U_t| \right\|_{\mathbb{L}^2} \quad \text{to } X.$$

Let $T > 0$ be arbitrary fixed. Let $t \in [0,T]$ and $(r,z) \in [0,T] \times [0,1]$ be fixed.

We show firstly by recurrence that for any $n \geqslant 0$, X_t^n is differentiable at point (r,z) in the Malliavin sense.

Recurrence Hypothesis.

(i) $X_{i,t}^n \in \mathbb{D}^{1,2}\ \forall t \in [0,T]\ \forall i = 1,\dots,d$.

(ii) $\sup_{t\in[0,T]} \sum_{l=1}^{d'} E(\int_0^\infty \int_0^1 |D_{(r,z)}^l X_t^n|^2 \, \mathrm{d}z\, \mathrm{d}r) < \infty$ where $|D_{(r,z)}^l X_t^n|^2 = \sum_{i=1}^d (D_{(r,z)}^l X_{i,t}^n)^2$.

For $n = 0$, Recurrence Hypothesis is satisfied.

We assume that it is true at rank n. According to Nualart (1995) Proposition 1.2.2, since σ and b are functions of class $\mathscr{C}_b^1$, we notice that $\forall i = 1,\dots,d\ \forall k = 1,\dots,d'$, $\sigma_{i,k}(X_t^n - Y_t(\alpha)) \in \mathbb{D}^{1,2}$ and $b_i(X_t^n - Y_t(\alpha)) \in \mathbb{D}^{1,2}$.

As for the Brownian Motion, we can show that derivative and integral commute (see Nualart, 1995), then $X_{i,t}^{n+1} \in \mathbb{D}^{1,2}\ \forall t \in [0,T]\ \forall i = 1,\dots,d$. Moreover, its derivative at point $(r,z) \in [0,T] \times [0,1]$ in relation to the lth white noise W_l is given by

$$\begin{aligned} D_{(r,z)}^l X_{i,t}^{n+1} &= \sigma_{i,l}(X_r^n - Y_r(z)) \\ &\quad + \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}(X_s^n - Y_s(\alpha)) D_{(r,z)}^l X_{m,s}^n W_k(\mathrm{d}\alpha, \mathrm{d}s) \\ &\quad + \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i(X_s^n - Y_s(\alpha)) D_{(r,z)}^l X_{m,s}^n \, \mathrm{d}\alpha\, \mathrm{d}s \end{aligned}$$

if $r \leqslant t$, and $D_{(r,z)}^l X_{i,t}^{n+1} = 0$ else.

We still have to check that

$$\sup_{t\in[0,T]} \sum_{l=1}^{d'} E\left[\int_0^\infty \int_0^1 |D_{(r,z)}^l X_t^{n+1}|^2 \, \mathrm{d}z\, \mathrm{d}r \right] < \infty.$$

We define

$$S_n(t) = \sum_{l=1}^{d'} E\left[\int_0^\infty \int_0^1 |D_{(r,z)}^l X_t^n|^2 \, \mathrm{d}z\, \mathrm{d}r \right] = \sum_{l=1}^{d'} E\left[\int_0^t \int_0^1 |D_{(r,z)}^l X_t^n|^2 \, \mathrm{d}z\, \mathrm{d}r \right].$$

According to Recurrence Hypothesis, $\sup_{t\in[0,T]} S_n(t) < \infty$. Let us study $\sup_{t\in[0,T]} S_{n+1}(t)$.

Let $l \in \{1,\dots,d'\}$ be arbitrary fixed. We divide in three parts the expectation $E[\int_0^t \int_0^1 |D_{(r,z)}^l X_{i,t}^{n+1}|^2 \, \mathrm{d}z\, \mathrm{d}r]$.

We define

$$E_1 = E\left[\int_0^t\int_0^1 |\sigma_{i,l}(X_r^n - Y_r(z))|^2\,\mathrm{d}z\,\mathrm{d}r\right]$$

$$\leqslant 4K_\sigma^2 E\left[\int_0^t\int_0^1 |X_r^n|^2 + |Y_r(z)|^2\,\mathrm{d}z\,\mathrm{d}r\right] + 2T|\sigma(0)|^2$$

since σ is Lipschitz continuous

$$\leqslant 4K_\sigma^2 T\left(\sup_n E\left[\sup_{0\leqslant r\leqslant T}|X_r^n|^2\right] + E_\alpha\left[\sup_{0\leqslant r\leqslant T}|Y_r|^2\right]\right) + 2T|\sigma(0)|^2$$

According to (13), we have $\sup_{0\leqslant t\leqslant T} E_1 < \infty$.

We define

$$E_2 = E\left[\int_0^t\int_0^1\left|\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}(X_s^n - Y_s(\alpha))D_{(r,z)}^l X_{m,s}^n W_k(\mathrm{d}\alpha,\mathrm{d}s)\right|^2\mathrm{d}z\,\mathrm{d}r\right]$$

$$= \int_0^t\int_0^1 E\left[\int_r^t\int_0^1\sum_{k=1}^{d'}\left(\sum_{m=1}^{d}\partial_m\sigma_{i,k}(X_s^n - Y_s(\alpha))D_{(r,z)}^l X_{m,s}^n\right)^2\mathrm{d}\alpha\,\mathrm{d}s\right]\mathrm{d}z\,\mathrm{d}r$$

since W_k are independent white noises

$$\leqslant d\sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1(\partial_m\sigma_{i,k}(X_s^n - Y_s(\alpha))D_{(r,z)}^l X_{m,s}^n)^2\,\mathrm{d}\alpha\,\mathrm{d}s\right]\mathrm{d}z\,\mathrm{d}r$$

using Hölder's Inequality.

Since the partial derivatives of σ are bounded by K_σ,

$$E_2 \leqslant dK_\sigma^2\sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1[D_{(r,z)}^l X_{m,s}^n]^2\,\mathrm{d}\alpha\,\mathrm{d}s\right]\mathrm{d}z\,\mathrm{d}r$$

$$= d'dK_\sigma^2\int_0^t E\left[\int_0^s\int_0^1|D_{(r,z)}^l X_s^n|^2\,\mathrm{d}z\,\mathrm{d}r\right]\mathrm{d}s$$

using Fubini's Theorem.

We consider now

$$E_3 = E\left[\int_0^t\int_0^1\left|\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i(X_s^n - Y_s(\alpha))D_{(r,z)}^l X_{m,s}^n\,\mathrm{d}\alpha\,\mathrm{d}s\right|^2\mathrm{d}z\,\mathrm{d}r\right].$$

Using the same method as for integral E_2, we finally prove that

$$S_{n+1}(t) \leqslant C_0 + C_1\int_0^t S_n(s)\,\mathrm{d}s \leqslant C_0 + C_1 T\sup_{0\leqslant t\leqslant T} S_n(t) < +\infty \tag{14}$$

with

$$C_0 = 12dd'K_\sigma^2 T\left(\sup_n E\left[\sup_{0\leqslant r\leqslant T}|X_r^n|^2\right] + E_\alpha\left[\sup_{0\leqslant r\leqslant T}|Y_r|^2\right]\right) + 6dd'T|\sigma(0)|^2,$$

$$C_1 = 6d^2\max(d'K_\sigma^2, K_b^2 T).$$

Thus, Recurrence Hypothesis is satisfied for any $n \geqslant 0$.

Since $S_0 = 0$, we notice that we have in fact a stronger result than property (ii). Estimate (14) implies that $\sup_{n\geqslant 0}\sup_{t\in[0,T]} S_n(t) \leqslant C_0 \mathrm{e}^{C_1 T}$. Finally, we have proved

$$\forall n \geqslant 0\ \forall t\in[0,T]\ \forall i = 1,\ldots,d\ X_{i,t}^n \in \mathbb{D}^{1,2}, \tag{15}$$

$$\sup_{n\geqslant 0}\sup_{t\in[0,T]}\sum_{l=1}^{d'} E\left(\int_0^t\int_0^1 |D_{(r,z)}^l X_t^n|^2\,\mathrm{d}z\,\mathrm{d}r\right) < \infty. \tag{16}$$

Since the sequence (X^n) converges uniformly on $[0,T]$ in $\mathbb{L}^2$ to X and thanks to (15) and (16), we deduce that X is differentiable (see Nualart, 1995, Lemma 1.2.3). Moreover, the sequence of derivatives (DX^n) converges to DX for the weak topology on $\mathbb{L}^2([0,T]\times[0,1]\times\Omega)$. Thus, the theorem is proved. □

2.2. The upper order derivatives

We state that X belongs to $\mathbb{D}^\infty$ under a stronger assumption on σ and b.

Assumption (H^∞). σ and b are Lipschitz continuous functions of class $\mathscr{C}^\infty$ with bounded derivatives from $\mathbb{R}^d$ to $\mathscr{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$, respectively.

Notations. Let $k \geqslant 1$. We define $\lambda_k = ((r_1,z_1),\ldots,(r_k,z_k))$ and

$$\hat{\lambda}_m = ((r_1,z_1),\ldots,(r_{m-1},z_{m-1}),(r_{m+1},z_{m+1}),\ldots,(r_k,z_k))$$

with $r_m \in [0,t]$ and $z_m \in [0,1]$ for $m = 1,\ldots,k$.

Let us now define $l(E) = l_{\varepsilon_1},\ldots,l_{\varepsilon_\eta}$ and $\lambda(E) = ((r_{\varepsilon_1},z_{\varepsilon_1}),\ldots,(r_{\varepsilon_\eta},z_{\varepsilon_\eta}))$ for any subset $E = \{\varepsilon_1,\ldots,\varepsilon_\eta\}$ of $\{1,\ldots,k\}$. We consider

$$\Sigma_{j,(l_1,\ldots,l_k)}^i((s,\alpha),\lambda_k) = \sum\sum_{k_1,\ldots,k_v=1}^{d} \partial_{k_1,\ldots,k_v}\sigma_{i,j}(X_s - Y_s(\alpha))D_{\lambda(E_1)}^{l(E_1)}X_{k_1,s}\cdots D_{\lambda(E_v)}^{l(E_v)}X_{k_v,s}$$

$$\beta_{(l_1,\ldots,l_k)}^i((s,\alpha),\lambda_k) = \sum\sum_{k_1,\ldots,k_v=1}^{d} \partial_{k_1,\ldots,k_v} b_i(X_s - Y_s(\alpha))D_{\lambda(E_1)}^{l(E_1)}X_{k_1,s}\cdots D_{\lambda(E_v)}^{l(E_v)}X_{k_v,s},$$

where the first sum is taken on all partitions $E_1\cup\cdots\cup E_v = \{1,\ldots,k\}$.

We define at last,

$$\Sigma_j^i((s,\alpha)) = \sigma_{ij}(X_s - Y_s(\alpha)).$$

We denote by $r_1\vee\cdots\vee r_k = \sup\{r_1,\ldots,r_k\}$.

Theorem 10. *Assume that* $X_0 \in \mathbb{L}^p$, *for any* $p \geqslant 1$. *Under Assumption* (H^∞), $\forall t \geqslant 0$ $X_t \in \mathbb{D}^\infty$. *Moreover, the* ith *component of one of its derivative of order* k *at point* $\lambda_k = ((r_1, z_1), \ldots, (r_k, z_k))$ *is given by the following equation*:

$$
\begin{aligned}
D^{l_1,\ldots,l_k}_{\lambda_k} X_{i,t} = {} & \sum_{m=1}^{k} \Sigma^i_{l_m,(l_1,\ldots l_{m-1},l_{m+1},\ldots,l_k)}((r_m, z_m), \hat{\lambda}_m) \\
& + \int_{r_1 \vee \cdots \vee r_k}^{t} \int_0^1 \sum_{j=1}^{d'} \Sigma^i_{j,(l_1,\ldots,l_k)}((s,\alpha), \lambda_k) W_j(\mathrm{d}\alpha, \mathrm{d}s) \\
& + \int_{r_1 \vee \cdots \vee r_k}^{t} \int_0^1 \beta^i_{(l_1,\ldots,l_k)}((s,\alpha), \lambda_k)\, \mathrm{d}\alpha\, \mathrm{d}s
\end{aligned} \tag{17}
$$

if $t \geqslant r_1 \vee \cdots \vee r_k$ *and* $D^{l_1,\ldots,l_k}_{\lambda_k} X_{i,t} = 0$ *if* $t < r_1 \vee \cdots \vee r_k$.

Remark 11. *In expression* (17) *of the* kth *derivative, the terms in the first sum with* $r_m < r_1 \vee \cdots \vee r_k$ *are equal to* 0.

Proof. We use again the Picard sequence $(X^n)_{n \geqslant 0}$ defined by (12). For any $p \geqslant 2$, $n \geqslant 0$, $X^n \in \mathbb{L}^p$ and (X^n) converges uniformly to X in $\mathbb{L}^p$. As σ and b satisfy Assumption (H^∞), using the same method as in the previous paragraph, we prove that $X^n_t \in \mathbb{D}^{1,p}$ $\forall p \geqslant 1$ for any $t \geqslant 0$. By recurrence, we prove that $\forall t \geqslant 0$, $\forall n \geqslant 0$ $X^n_t \in \mathbb{D}^\infty$.

Let us fix $T > 0$.

Recurrence Hypothesis (h_n):

(i) $X^n_{i,t} \in \mathbb{D}^\infty$, $\forall t \in [0, T]$, $\forall i = 1, \ldots, d$.

(ii) $\sup_{t \in [0,T]} \sum_{l_1,\ldots,l_k=1}^{d'} E(\int_{([0,t]\times[0,1])^k} |D^{l_1,\ldots,l_k}_{\lambda_k} X^n_{i,t}|^p \, \mathrm{d}\lambda_k) < \infty$ $\forall p \geqslant 1, \forall k \geqslant 1$.

(iii) the derivatives of order k have the following expression:

$$
\begin{aligned}
D^{l_1,\ldots,l_k}_{\lambda_k} X^{n+1}_{i,t} = {} & \sum_{m=1}^{k} \Sigma^{n,i}_{l_m,(l_1,\ldots l_{m-1},l_{m+1},\ldots,l_k)}((r_m, z_m), \hat{\lambda}_m) \\
& + \int_{r_1 \vee \cdots \vee r_k}^{t} \int_0^1 \sum_{j=1}^{d'} \Sigma^{n,i}_{j,(l_1,\ldots,l_k)}((s,\alpha), \lambda_k) W_j(\mathrm{d}\alpha, \mathrm{d}s) \\
& + \int_{r_1 \vee \cdots \vee r_k}^{t} \int_0^1 \beta^{n,i}_{(l_1,\ldots,l_k)}((s,\alpha), \lambda_k)\, \mathrm{d}\alpha\, \mathrm{d}s
\end{aligned} \tag{18}
$$

if $t \geqslant r_1 \vee \cdots \vee r_k$ and $D^{l_1,\ldots,l_k}_{\lambda_k} X^n_{i,t} = 0$ else, where Σ^n and β^n are defined as Σ and β replacing X with X^n.

Hypothesis (h_0) is satisfied.

Let us assume that Hypothesis (h_n) is true, and let us study (h_{n+1}). According to Assumption (H^∞) and adapting the computation of the first derivative, it is easy to state that the two first properties are satisfied. We just check the expression of the kth derivative by recurrence on k.

For $k=1$, we have

$$D^{l}_{(r,z)}X^{n+1}_{i,t} = \Sigma^{n,i}_{l}((r,z)) + \int_r^t \int_0^1 \sum_{j=1}^{d'} \Sigma^{n,i}_{j,(l)}((s,\alpha),(r,z))W_j(\mathrm{d}\alpha,\mathrm{d}s)$$

$$+ \int_r^t \int_0^1 \beta^{n,i}_{(l)}((s,\alpha),(r,z))\,\mathrm{d}\alpha\,\mathrm{d}s,$$

then expression (18) is satisfied.

We assume that the expression (18) of the kth derivative is true, and we now compute the derivative of order $k+1$

$$\begin{aligned}
&D^{l_{k+1}}_{(r_{k+1},z_{k+1})}(D^{l_1,\dots,l_k}_{\lambda_k}X^{n+1}_{i,t})\\
&\quad = D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\sum_{m=1}^{k} \Sigma^{n,i}_{l_m,(l_1,\dots l_{m-1},l_{m+1},\dots,l_k)}((r_m,z_m),\hat{\lambda}_m)\right)\\
&\qquad + \Sigma^{n,i}_{l_{k+1},(l_1,\dots,l_k)}((r_{k+1},z_{k+1}),\lambda_k)\\
&\qquad + \int_{r_1\vee\cdots\vee r_k\vee r_{k+1}}^t \int_0^1 \sum_{j=1}^{d'} D^{l_{k+1}}_{(r_{k+1},z_{k+1})}(\Sigma^{n,i}_{j,(l_1,\dots,l_k)}((s,\alpha),\lambda_k))W_j(\mathrm{d}\alpha,\mathrm{d}s)\\
&\qquad + \int_{r_1\vee\cdots\vee r_k\vee r_{k+1}}^t \int_0^1 D^{l_{k+1}}_{(r_{k+1},z_{k+1})}(\beta^{n,i}_{(l_1,\dots,l_k)}((s,\alpha),\lambda_k))\,\mathrm{d}\alpha\,\mathrm{d}s.
\end{aligned}$$

Using some elementary computations, we obtain

$$\begin{aligned}
D^{l_{k+1}}_{(r_{k+1},z_{k+1})}(D^{l_1,\dots,l_k}_{\lambda_k}X^{n+1}_{i,t}) &= \sum_{m=1}^{k} \Sigma^{n,i}_{l_m,(l_1,\dots l_{m-1},l_{m+1},\dots,l_k,l_{k+1})}((r_m,z_m),\hat{\lambda}_m)\\
&\quad + \Sigma^{n,i}_{l_{k+1},(l_1,\dots,l_k)}((r_{k+1},z_{k+1}),\hat{\lambda}_{k+1})\\
&\quad + \int_{r_1\vee\cdots\vee r_{k+1}}^t \int_0^1 \sum_{j=1}^{d'} \Sigma^{n,i}_{j,(l_1,\dots,l_k,l_{k+1})}((s,\alpha),\lambda_{k+1})W_j(\mathrm{d}\alpha,\mathrm{d}s)\\
&\quad + \int_{r_1\vee\cdots\vee r_{k+1}}^t \int_0^1 \beta^{n,i}_{(l_1,\dots,l_k,l_{k+1})}((s,\alpha),\lambda_k)\,\mathrm{d}\alpha\,\mathrm{d}s.
\end{aligned}$$

So by recurrence, property (iii) of (h_{n+1}) is proved and consequently for any $n \geqslant 0$ Recurrence Hypothesis (h_n) is satisfied.

Moreover, as in the computation of the first derivative, we have a stronger property than (ii) in (h_n):

Lemma 12. *If we denote by*

$$S_{n,k}(t) = \sum_{l_1,\dots,l_k=1}^{d'} E\left(\int_{([0,t]\times[0,1])^k} |D_{\lambda_k}^{l_1,\dots,l_k} X_t^n|^p \,\mathrm{d}\lambda_k\right),$$

$$M_k = \sup_{0\leqslant q\leqslant k}\ \sup_{n\geqslant 0}\ \sup_{t\in[0,T]} S_{n,q}(t),$$

then for any $k \geqslant 1$, $M_k < \infty$.

Proof. The proof is similar to the proof of Theorem 9. □

As (X^n) converges to X in $\mathbb{L}^p$ uniformly on $[0,T]$ for any $T > 0$, the process X satisfies the conditions of Lemma 1.5.4 in Nualart (2000). Then, the theorem is proved. □

3. Existence of a weak function-solution of the Landau equation

Under some suitable conditions on the function h, the Landau coefficients satisfy Assumption (H^1) (see Remark 8). Consequently, if X_0 belongs to $\mathbb{L}^2$, the process X solution of (NSDE(σ, b)) is differentiable in the Malliavin sense. Let us now study the Malliavin matrix $I_t = \int_0^T \int_0^1 D_{(r,z)}X_t.(D_{(r,z)}X_t)^* \,\mathrm{d}z\,\mathrm{d}r$ for any $t > 0$ to state the following theorem.

Theorem 13. *Assume that* X_0 *is a* $\mathbb{R}^d$*-valued random vector with a finite* 2*-order moment. Let* σ *and* b *be the coefficients of the Landau equation defined, respectively, by* (7), (2) *and* (4). *We assume that* σ *and* b *are Lipschitz continuous of class* $\mathscr{C}^1$. *If the distribution of* X_0 *is not a Dirac mass and if we denote by* (X, Y) *the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)) W^{d'}(\mathrm{d}\alpha, \mathrm{d}s)$$
$$+ \int_0^t \int_0^1 b(X_s - Y_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s \quad (\mathrm{NSDE}(\sigma, b))$$

then, for any $t > 0$ *the distribution* P_t *of* X_t *is absolutely continuous with respect the Lebesgue measure.*

Corollary 14. *Let* P_0 *be a probability measure such that* $\int |x|^2 P_0(\mathrm{d}x) < \infty$. *Let* σ *and* b *be the coefficients of the Landau equation defined, respectively, by* (7), (2) *and* (4). *We assume that* σ *and* b *are Lipschitz continuous of class* $\mathscr{C}^1$. *If* P_0 *is not a Dirac measure, there exists a unique weak function-solution of the Landau equation with initial data* P_0.

Proof. Let X_0 be a random vector with distribution P_0 and X be a solution of (NSDE(σ, b)) with initial data X_0. If we denote by f_t the density of the distribution of

X_t, then, using Itô's Formula, the function f, defined by $f(x,t)=f_t(x)$ for $t>0$, is a weak function solution of Landau equation (5) with initial data P_0.

The uniqueness is given by Corollary 7. $\square$

Remark 15. *Without any restriction, we can assume that* $\mathbf{E}[\mathbf{X_0}]=\mathbf{0}$ *to simplify the computations.*

Proof. By conservation of momentum, if we define for any $t \geqslant 0$, $X'_t = X_t - E[X_0]$, the expectation of X' is equal to 0 and X' satisfies the following equation:

$$X'_t = X'_0 + \int_0^t \int_0^1 \sigma(X'_s - Y'_s(\alpha))W^{d'}(\mathrm{d}\alpha, \mathrm{d}s) + \int_0^t \int_0^1 b(X'_s - Y'_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s$$

with $Y'_s(\alpha) = Y_s(\alpha) - E[X_0]$.

As $\mathscr{L}(X) = \mathscr{L}_\alpha(Y)$, we also have $\mathscr{L}(X') = \mathscr{L}_\alpha(Y')$.

If we prove that the distribution of X'_t has a density f'_t with respect to the Lebesgue measure, then X_t has a density given by $f_t(z) = f'_t(z - E[X_0])$. $\square$

Proof of Theorem 13. We recall the expression of the first Malliavin derivative of X at point $(r,z) \in [0,\infty) \times [0,1]$:

$$\begin{aligned} D^l_{(r,z)}X_{i,t} &= \sigma_{i,l}(X_r - Y_r(z)) \\ &\quad + \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}(X_s - Y_s(\alpha)) D^l_{(r,z)}X_{m,s} W_k(\mathrm{d}\alpha, \mathrm{d}s) \\ &\quad + \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i(X_s - Y_s(\alpha)) D^l_{(r,z)}X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s \end{aligned}$$

if $t \geqslant r$ and $D^l_{(r,z)}X_{i,t} = 0$ else.

We fix $(r,z) \in [0,\infty) \times [0,1]$ and we define

$$S_k(.) = (\partial_m \sigma_{i,k}(.))_{1 \leqslant i,m \leqslant d}$$

$$B(.) = (\partial_m b_i(.))_{1 \leqslant i,m \leqslant d}$$

Thus we give a matricial expression of the derivative of X

$$\begin{aligned} D_{(r,z)}X_t &\underset{\text{if } t \geqslant r}{=} \sigma(X_r - Y_r(z)) + \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k(X_s - Y_s(\alpha)) D_{(r,z)}X_s W_k(\mathrm{d}\alpha, \mathrm{d}s) \\ &\quad + \int_r^t \int_0^1 B(X_s - Y_s(\alpha)) D_{(r,z)}X_s\,\mathrm{d}\alpha\,\mathrm{d}s \\ D_{(r,z)}X_t &\underset{\text{if } t < r}{=} 0 \end{aligned}$$

Let us define the semimartingale Z^r, for any $t \geqslant r$,

$$Z_t^r = \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k(X_s - Y_s(\alpha)) W_k\,(\mathrm{d}\alpha, \mathrm{d}s) + \int_r^t \int_0^1 B(X_s - Y_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s$$

As S and B are bounded, $(Z_t^r)_{t \geqslant r}$ is a continuous semimartingale and the first derivative satisfies the equation

$$D_{(r,z)}X_t = \sigma(X_r - Y_r(z)) + \int_r^t \mathrm{d}Z_s^r . D_{(r,z)}X_s \tag{19}$$

Using the results of Jacod (1982), there is a unique solution of (19) defined almost surely, for any $t \geqslant r$ by

$$D_{(r,z)}X_t = \mathscr{E}(Z)_t^r . \sigma(X_r - Y_r(z))$$

with $\mathscr{E}(Z)_t^r$ invertible for any $t \geqslant r$.

We fix now $t > 0$.

We want to apply Theorem (a), thus study if the Malliavin matrix I_t is invertible a.s.

$$\begin{aligned} I_t &= \int_0^\infty \int_0^1 D_{(r,z)}X_t . (D_{(r,z)}X_t)^*\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_0^t \int_0^1 D_{(r,z)}X_t . (D_{(r,z)}X_t)^*\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_0^t \int_0^1 \mathscr{E}(Z)_t^r . \sigma(X_r - Y_r(z)) . \sigma^*(X_r - Y_r(z)) . (\mathscr{E}(Z)_t^r)^*\,\mathrm{d}r\,\mathrm{d}z \\ &= \int_0^t \mathscr{E}(Z)_t^r . \left(\int_0^1 a(X_r - Y_r(z))\,\mathrm{d}z \right) . (\mathscr{E}(Z)_t^r)^*\,\mathrm{d}r, \end{aligned}$$

I_t is a nonnegative symmetric matrix, then I_t is invertible if and only if $V^*.I_t.V > 0$ for any $V \in \mathbb{R}^d \setminus \{0\}$.

We define $\Gamma_r = \int_0^1 a(X_r - Y_r(z))\,\mathrm{d}z$.

We prove the theorem by contradiction.

Assumption. Let us suppose that I_t is not "invertible a.s.".

Then, there exists a subset $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) > 0$, such that $\forall \omega \in \Omega_1$ $I_t(\omega)$ is not invertible.

Let Ω_2 be such that $\mathbb{P}(\Omega_2) = 1$ and $\forall \omega \in \Omega_2$ $\forall r \leqslant t$, $\mathscr{E}(Z)_t^r(\omega)$ is invertible. We define $\Omega_0 = \Omega_1 \cap \Omega_2$, and we notice that $\mathbb{P}(\Omega_0) > 0$.

Let us fix $\omega_0 \in \Omega_0$.

As $I_t(\omega_0)$ is not invertible, there exists a vector $V = V(\omega_0, t) \in \mathbb{R}^d \setminus \{0\}$ such that

$$V^* . I_{\omega_0}(t) . V = \int_0^t V^* . \mathscr{E}(Z)_t^r(\omega_0) . \Gamma_r(\omega_0) . (\mathscr{E}(Z)_t^r(\omega_0))^* . V\,\mathrm{d}r = 0$$

As for any $r \leqslant t$, $\mathscr{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathscr{E}(Z)_t^r(\omega_0))^*$ is a nonnegative symmetric matrix, we notice that

$$V^*.\mathscr{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathscr{E}(Z)_t^r(\omega_0))^*.V \geqslant 0.$$

Then, on a subset J_{ω_0} of full measure in $[0,t]$, $V^*.\mathscr{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathscr{E}(Z)_t^r(\omega_0))^*.V = 0$, which implies that $\forall r \in J_{\omega_0}$, $\mathscr{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathscr{E}(Z)_t^r(\omega_0))^*$ is a noninvertible matrix. However, since $\Omega_0 \subset \Omega_2$, $\mathscr{E}(Z)_t^r(\omega_0)$ is invertible for any $r \leqslant t$, and consequently $\Gamma_r(\omega_0)$ is not invertible for $r \in J_{\omega_0}$.

Let us now study if the situation "$\Gamma_r(\omega_0)$ noninvertible for almost all r" is possible.

Using Lebesgue's Theorem, we notice that the mapping $r \to \Gamma_r(\omega_0)$ is continuous. Consequently, $\Gamma_r(\omega_0)$ noninvertible for almost all r implies that $\Gamma_r(\omega_0)$ is noninvertible for any $r \in [0,t]$.

Let $V = (V_i)_{1 \leqslant i \leqslant d}$ be a vector in $\mathbb{R}^d \setminus \{0\}$.

Using lower bound (3) of h, since $E[X_t] = 0$ and $\mathscr{L}(X_t) = \mathscr{L}_\alpha(Y_t)$ $\forall t \geqslant 0$, we obtain the following estimate:

$$\begin{aligned}
&V^*.\Gamma_r(\omega_0).V \\
&\quad = \int_0^1 h(|X_r(\omega_0) - Y_r(z)|^2)\,\Bigg[|V|^2|X_r(\omega_0) - Y_r(z)|^2 \\
&\qquad - \left(\sum_{i=1}^d V_i(X_{i,r}(\omega_0) - Y_{i,r}(z))\right)^2\Bigg]\,\mathrm{d}z \\
&\quad \geqslant m\left[|V|^2|X_r(\omega_0)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega_0)\right)^2 + E(|X_r|^2|V|^2 - \langle X_r, V\rangle^2)\right]
\end{aligned}$$

Using Cauchy–Schwarz's inequality, we notice that

$$|V|^2|X_r(\omega_0)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega_0)\right)^2 \geqslant 0$$

and consequently,

$$V^*.\Gamma_r(\omega_0).V \geqslant mE(|X_r|^2|V|^2 - \langle X_r, V\rangle^2), \tag{20}$$

$\Gamma_r(\omega_0)$ noninvertible means that for any $r \in [0,t]$ there exists $V_r = V_r(\omega_0) \in \mathbb{R}^d \setminus \{0\}$ such that $V_r^*.\Gamma_r(\omega_0).V_r = 0$. We can assume that $|V_r| = 1$ without restriction. Nevertheless, using expression (20), this implies that there is equality in Cauchy–Schwarz, i.e. $\forall r \in [0,t]$ there exists a random variable U_r such that for any $\omega \in \Omega$,

$$X_r(\omega) = U_r(\omega)V_r. \tag{21}$$

Using the conservation of the momentum, of the kinetic energy and $|V_r| = 1$, we notice that $E[U_r] = E[X_0] = 0$ and $E[U_r^2] = E[|X_0|^2]$. Moreover, since the distribution of X_0 is not a Dirac measure, $E[U_r^2] \neq 0$ for any $r \in [0,t]$.

The distribution of a solution of (NSDE(σ,b)) is a measure-solution of the Landau equation (6). Then, we will now study if the distribution of a process defined by (21) can be a solution of the Landau equation.

We denote by Q the distribution of the process U and by Q_t the distribution on $\mathbb{R}$ of U_t. Using (21), Eq. (6) writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\varphi(xV_t)Q_t(\mathrm{d}x)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}\times\mathbb{R}}a_{ij}((x-y)V_t)\partial_{ij}\varphi(xV_t)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y)$$
$$+\sum_{i=1}^{d}\int_{\mathbb{R}\times\mathbb{R}}b_i((x-y)V_t)\partial_i\varphi(xV_t)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y)$$

for any test function $\varphi\in\mathscr{C}_b^2(\mathbb{R}^d,\mathbb{R})$.

As $|V_t|=1$, for $i,j\in\{1,\dots,d\}$,

$$a_{ij}((x-y)V_t)=(x-y)^2h((x-y)^2)(\delta_{ij}-V_{i,t}V_{j,t}),$$
$$b_i((x-y)V_t)=-(d-1)(x-y)h((x-y)^2)V_{i,t}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\varphi(xV_t)Q_t(\mathrm{d}x)$$
$$=\frac{1}{2}\sum_{i,j=1}^{d}(\delta_{ij}-V_{i,t}V_{j,t})\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2h((x-y)^2)\partial_{ij}\varphi(xV_t)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y)$$
$$-(d-1)\sum_{i=1}^{d}V_{i,t}\int_{\mathbb{R}\times\mathbb{R}}(x-y)h((x-y)^2)\partial_i\varphi(xV_t)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y).$$

We now explicit the equation satisfied by the 2-order moments of X: let $k,l\in\mathbb{N}$, $k\neq l$. Using $\varphi(v)=v_k^2$ or $\varphi(v)=v_kv_l$, $v\in\mathbb{R}^d$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}V_{k,t}^2x^2Q_t(\mathrm{d}x)=E[|X_0|^2]\frac{\mathrm{d}}{\mathrm{d}t}V_{k,t}^2$$
$$=(1-dV_{k,t}^2)\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2h((x-y)^2)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}V_{k,t}V_{l,t}x^2Q_t(\mathrm{d}x)=E[|X_0|^2]\frac{\mathrm{d}}{\mathrm{d}t}V_{k,t}V_{l,t}$$
$$=-dV_{k,t}V_{l,t}\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2h((x-y)^2)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y).$$

Let us define $f(t)=\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2h((x-y)^2)Q_t(\mathrm{d}x)Q_t(\mathrm{d}y)$. As h satisfies (3) and $E[|U_t|^2]\neq 0$, for any $t\geqslant 0$, we notice that $f(t)>0$.

Let us now compute $(\mathrm{d}/\mathrm{d}t)(V_{k,t}^2V_{l,t}^2)$ using two different ways:

$$E[|X_0|^2]\frac{\mathrm{d}}{\mathrm{d}t}(V_{k,t}^2V_{l,t}^2)=V_{k,t}^2E[|X_0|^2]\frac{\mathrm{d}}{\mathrm{d}t}V_{l,t}^2+V_{l,t}^2E[|X_0|^2]\frac{\mathrm{d}}{\mathrm{d}t}V_{k,t}^2$$
$$=V_{k,t}^2f(t)+V_{l,t}^2f(t)-2dV_{k,t}^2V_{l,t}^2f(t),$$

$$E[|X_0|^2]\,\frac{\mathrm{d}}{\mathrm{d}t}(V_{k,t}^2V_{l,t}^2) = 2V_{k,t}V_{l,t}E[|X_0|^2]\,\frac{\mathrm{d}}{\mathrm{d}t}(V_{k,t}V_{l,t})$$
$$= -2dV_{k,t}^2V_{l,t}^2f(t).$$

Then $V_t = 0$ which is impossible.

Finally, I_t is invertible a.s. for any $t > 0$ and according to Theorem (a), the theorem is proved. □

Remark 16. *We notice that the matrix* $\Gamma_r = \int_0^1 a(X_r - Y_r(z))\,\mathrm{d}z$ *is invertible a.s., whereas* $\det(a(X_r - Y_r(z))) = 0$ *for any* r, z. *In fact, thanks to the nonlinearity of equation* (NSDE(σ, b)), *we can conclude that the Malliavin matrix has an inverse a.s.*

Remark 17. *A consequence of Theorem* 13 *is*

$$E(|X_t|^2|V|^2 - \langle X_t, V\rangle^2) > 0$$

for any $V \in \mathbb{R}^d \setminus \{0\}$, *for any* $t > 0$, *when the random vector* X_0 *is not a constant.*

4. Regularity of the weak function-solution

Theorem 18. *Let* X_0 *be a random vector such that* $E[|X_0|^p] < \infty$ *for any* $p \geqslant 1$. *Let* σ *and* b *be the coefficients of the Landau equation, respectively, defined by* (7), (2) *and* (4). *We assume that* σ *and* b *are Lipschitz continuous and infinitely differentiable with bounded derivatives. If the distribution of* X_0 *is not a Dirac mass and if we denote by* X *the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)).W^{d'}(\mathrm{d}\alpha, \mathrm{d}s)$$
$$+ \int_0^t \int_0^1 b(X_s - Y_s(\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s \quad (\mathrm{NSDE}(\sigma, b))$$

then for any $t > 0$ *the distribution of* X_t *has a bounded density of class* $\mathscr{C}^\infty$ *with bounded derivatives with respect to the Lebesgue measure on* $\mathbb{R}^d$.

Corollary 19. *Let* P_0 *be a probability measure such that* $\int |x|^p P_0(\mathrm{d}x) < \infty$ *for any* $p \geqslant 1$. *Let* σ *and* b *be the coefficients of the Landau equation defined, respectively, by* (7), (2) *and* (4). *We assume that* σ *and* b *are Lipschitz continuous and infinitely differentiable with bounded derivatives. If* P_0 *is not a Dirac measure, there exists a unique weak function solution of the Landau equation with initial data* P_0 *which is moreover of class* $\mathscr{C}^\infty$, *bounded on* $\mathbb{R}^d$ *and with bounded derivatives.*

Remark 20. *Using expressions* (10) *or* (11), *we notice that, if* h *is a bounded function of class such that* $h^{(l)}(x) = \mathrm{O}(1/x^{l+1})$ *when* $x \to +\infty$ *for any* $l \geqslant 1$, σ *and* b *are Lipschitz continuous functions of class* $\mathscr{C}^\infty$ *with bounded derivatives.*

Proof. As in the previous part, we assume that $E[X_0]=0$ to simplify the computations. As σ and b satisfy Assumption (H^∞), the process X is infinitely differentiable in the Malliavin sense. We need to study the moments of the inverse of the determinant of the Malliavin matrix I_t at time t, for any $t>0$, to apply Theorem (b). The expression of the determinant is complex, nevertheless we can notice that in dimension d,

$$(\det I_t)^{1/d} \geqslant \inf_{|V|=1} \langle I_t V, V\rangle,$$

where $\langle .,. \rangle$ is the euclidean scalar product in $\mathbb{R}^d$.

Moreover, see Bally et al. (1994) Lemma 3.4, property (ii) of Theorem (b) is satisfied as soon as for any $k\in\mathbb{N}$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-k}\mathbb{P}((\det I_t)^{1/d} < c\varepsilon) = 0, \tag{22}$$

where c is a positive constant which will be computed later.

Let $t>0$ be fixed.

As $(\det I_t)^{1/d} \geqslant \inf_{|V|=1}\langle I_t V, V\rangle$, we want to find a lower bound for $\inf_{|V|=1}\langle I_t V, V\rangle$.

Let ε be such that $0<\varepsilon<t/2$. We consider $V=(V_i)_{1\leqslant i\leqslant d}\in\mathbb{R}^d$ such that $|V|=1$.

$$\begin{aligned}
\langle I_t V, V\rangle &= \int_0^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^d D^l_{(r,z)} X_{i,t} V_i\right)^2 \mathrm{d}z\,\mathrm{d}r \\
&\geqslant \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^d D^l_{(r,z)} X_{i,t} V_i\right)^2 \mathrm{d}z\,\mathrm{d}r \\
&\geqslant \frac{2}{3} I_1 - 2I_2
\end{aligned}$$

with

$$I_1 = \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^d \sigma_{i,l}(X_r - Y_r(z))V_i\right)^2 \mathrm{d}z\,\mathrm{d}r$$

$$\begin{aligned}
I_2 = \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \Bigg[&\sum_{i=1}^d V_i \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^d \partial_m \sigma_{i,k}(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s} W_k(\mathrm{d}\alpha, \mathrm{d}s) \\
&+ \sum_{i=1}^d V_i \int_r^t \int_0^1 \sum_{m=1}^d \partial_m b_i(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s \Bigg]^2 \mathrm{d}z\,\mathrm{d}r
\end{aligned}$$

Then

$$\inf_{|V|=1} \langle I_t V, V\rangle \geqslant \frac{2}{3} \inf_{|V|=1} I_1 - 2 \sup_{|V|=1} I_2.$$

We want to minimize the first integral:

$$
\begin{aligned}
I_1 &= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l=1}^{d'}\left(\sum_{i=1}^{d}\sigma_{i,l}(X_r - Y_r(z))V_i\right)^2 \mathrm{d}z\,\mathrm{d}r\\
&= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{i,j=1}^{d} a_{i,j}(X_r - Y_r(z))V_iV_j\,\mathrm{d}z\,\mathrm{d}r\\
&= \int_{t-\varepsilon}^{t} V^*.\Gamma_r.V\,\mathrm{d}r.
\end{aligned}
$$

Using the results of Section 3, we obtain

$$
I_1 \geqslant m\int_{t-\varepsilon}^{t} E(|X_r|^2|V|^2 - \langle X_r, V\rangle^2)\,\mathrm{d}r.
$$

We define the function $f(V,r)=E(|X_r|^2|V|^2 - \langle X_r, V\rangle^2)$. We notice that f is a positive continuous function (see Remark 17) on the compact subset $D=\{V\in\mathbb{R}^d\colon |V|=1\}\times\{r\colon t/2\leqslant r\leqslant t\}$, then f reaches its minimum. So, if we denote by

$$
\tilde{c} = \inf\left\{f(V,r)\colon |V|=1 \text{ and } \frac{t}{2}\leqslant r\leqslant t\right\},
$$

we notice that $\tilde{c}$ is independent of $\omega\in\Omega$, $\tilde{c}>0$ and

$$
I_1 \geqslant m.\tilde{c}.\varepsilon.
$$

Let us now study $E[\sup_{|V|=1} I_2^p]$ for $p\geqslant 1$.

$$
\begin{aligned}
I_2 = \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l=1}^{d'}\Bigg[&\sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}(X_s - Y_s(\alpha))D^l_{(r,z)}X_{m,s}W_k(\mathrm{d}\alpha,\mathrm{d}s)\\
&+\sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i(X_s - Y_s(\alpha))D^l_{(r,z)}X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s\Bigg]^2\mathrm{d}z\,\mathrm{d}r.
\end{aligned}
$$

Using Burkholder–Davis–Gundy's and Hölder's inequalities, and the fact that $|V|=1$, we notice that

$$
\begin{aligned}
&E\left[\sup_{|V|=1} I_2^p\right]\\
&\quad\leqslant K\varepsilon^{p-1}\Bigg\{\int_{t-\varepsilon}^{t}\int_0^1\sum_{l,i,m}E\Bigg[\Bigg|\sum_{k=1}^{d'}\int_r^t\int_0^1\partial_m\sigma_{i,k}(X_s - Y_s(\alpha))\\
&\qquad\times D^l_{(r,z)}X_{m,s}W_k(\mathrm{d}\alpha,\mathrm{d}s)\Bigg|^{2p}\Bigg]\,\mathrm{d}z\,\mathrm{d}r
\end{aligned}
$$

$$+\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,m} E\left[\left|\int_r^t \partial_m b_i(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s\right|^{2p}\right]\mathrm{d}z\,\mathrm{d}r\right\}$$

$$\leqslant K\varepsilon^{p-1}\left\{\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,k,m} E\left[\left|\int_r^t \int_0^1 (\partial_m \sigma_{i,k}(X_s - Y_s(\alpha))\right.\right.\right.$$

$$\left.\left. D^l_{(r,z)} X_{m,s})^2\,\mathrm{d}\alpha\,\mathrm{d}s\right|^{p}\right]\mathrm{d}z\,\mathrm{d}r$$

$$\left.+\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,m} E\left[\left|\int_r^t \partial_m b_i(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s}\,\mathrm{d}\alpha\,\mathrm{d}s\right|^{2p}\right]\mathrm{d}z\,\mathrm{d}r\right\}.$$

As the derivatives of σ and b are bounded, using Hölder's inequality, we obtain

$$E\left[\sup_{|V|=1} I_2^p\right] \leqslant K\varepsilon^{2p-2}\left\{\int_{t-\varepsilon}^{t}\int_0^1 \sum_l E\left[\int_r^t |D^l_{(r,z)} X_s|^{2p}\,\mathrm{d}s\right]\mathrm{d}z\,\mathrm{d}r\right\}$$

using Fubini's Theorem

$$\leqslant K\varepsilon^{2p-2}\int_{t-\varepsilon}^{t} E\left[\int_{t-\varepsilon}^{s}\int_0^1 |D_{(r,z)} X_s|^{2p}\,\mathrm{d}z\,\mathrm{d}r\right]\mathrm{d}s.$$

Then, for any $p \geqslant 2$ there exists a constant $K = K(p,d,d',t)$ such that

$$E\left[\sup_{|V|=1} I_2^p\right] \leqslant K\varepsilon^{2p-1}\sup_{0\leqslant s\leqslant t} E\left[\int_0^s \int_0^1 |D_{(r,z)} X_s|^{2p}\,\mathrm{d}z\,\mathrm{d}r\right].$$

Let us now check that $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}((\det I_t)^{1/d} < c\varepsilon) = 0$, where $k \in \mathbb{N}$ is fixed and $c = \frac{1}{3} m\tilde{c}$ with $\tilde{c}$ the constant built in the study of the first integral I_1.

Let $p \in \mathbb{N}$ such that $p > k+1$.

$$\mathbb{P}((\det I_t)^{1/d} < c\varepsilon) \leqslant \mathbb{P}\left(\inf_{|V|=1}\langle I_t V, V\rangle < c\varepsilon\right)$$

$$\leqslant \mathbb{P}\left(\frac{2}{3}\inf_{|V|=1} I_1 - 2\sup_{|V|=1} I_2 < c\varepsilon\right)$$

$$\leqslant \mathbb{P}\left(\sup_{|V|=1} I_2 > \frac{c\varepsilon}{2}\right)$$

using Tchebychev's Inequality

$$\leqslant \left(\frac{2}{c}\right)^p \varepsilon^{-p} E\left[\sup_{|V|=1} I_2^p\right]$$

$$\leqslant K\varepsilon^{p-1}.$$

Thus, $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}((\det I_t)^{1/d} < c\varepsilon)=0$. We can apply Theorem (b), then there exists a density $f_t(v)$ to the distribution $P_t(\mathrm{d}v)$ for any $t>0$.

Using Nualart (1995) Lemma 2.1.5, the density $f_t(v)$ is given by

$$f_t(v)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}\langle x,v\rangle}\,\hat{v}_t(x)\,\mathrm{d}x,$$

where $\hat{v}_t(x)=\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\langle x,y\rangle}P_t(\mathrm{d}y)$ is the Fourier transform of P_t. Thanks to Theorem (b), we have the following estimates:

$$|x_1^{\alpha_1}\cdots x_d^{\alpha_d}\,\hat{v}_t(x)|\leqslant C_{\alpha,t}$$

for any $\alpha=\{\alpha_1,\ldots,\alpha_d\}\in\mathbb{N}$ with $C_{\alpha,t}$ depending on the moments of $(\det I_t)^{-1}$ and on the moments of the derivatives of X. Then f_t is a bounded function on $\mathbb{R}^d$ by

$$f_t(v)\leqslant C_{\{2,\ldots,2\},t}\int_{\mathbb{R}^d}\left(1\bigwedge_{i=1}^{d}\frac{1}{x_i^2}\right)\mathrm{d}x$$

and its derivatives have similar estimates. $\square$

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